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ON LOW-DIMENSIONAL FACES THAT HIGH-DIMENSIONAL POLYTOPES MUST HAVE

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We prove that every five-dimensional polytope has a two-dimensional face which is a triangle or a quadrilateral. We state and discuss the following conjecture: For every integer $k \geq 1$ there is an integer $f(k)$ such that every d-polytope, $d \geq f(k)$, has a k-dimensional face which is either a simplex or combinatorially isomorphic to the k-dimensional cube.

We give some related results concerning facet-forming polytopes and tilings. For example, sharpening a result of Schulte [25] we prove that there is no face to face tiling of $\mathbf{R}^5$ with crosspolytopes.

1. Introduction

A well-known consequence of the Euler relation for 3-dimensional convex polytopes (briefly, 3-polytopes) is:

Theorem 0. *Every 3-polytope has a face which is a triangle, a quadrilateral or a pentagon.*

In fact, the average number of vertices of the (two-dimensional) faces of a 3-polytope is strictly less than six. In a dual form, Theorem 0 asserts that the graph of every 3-polytope has a vertex of degree 3, 4 or 5.

The dodecahedron and the 120-cell (see, [10]) are examples of 3- and 4- dimensional polytopes all whose 2-faces are pentagons. Perles and Shephard asked in 1967 [22] whether similar examples exist in higher dimensions. Danzer [11] asked more generally whether there are d-polytopes, $d \geq 5$, with all 2-faces having at least five vertices. The existence of such polytopes (more precisely, their duals) is related to certain constructions of hyperbolic groups, see Gromov [14].

Theorem 1. *Every 5-polytope has a 2-face with three or four vertices.*

Theorem 1 clearly implies that every d-polytope, $d \geq 5$, has a 2-face with three or four vertices.

The proof of Theorem 1 relies on recent progress on face numbers and flag numbers of polytopes. In a similar way we prove

Theorem 2. *Every 5-polytope has a 3-face with a 3- or 4-valent vertex.*

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A d-polytope P is *facet-forming* if there is a $(d+1)$-polytope Q all whose facets ($=d$-faces) are combinatorially isomorphic to P. If no such Q exists, P is called a *non-facet*. For example, an n-gon is facet-forming if and only if $n \leq 5$. For some results on facet-forming polytopes and non-facets the reader is referred to [22, 2, 4, 23, 24].

Sharpening results of Perles and Shephard [22] we prove:

Theorem 3. *The d-crosspolytope, $d \geq 4$, the 24-cell, the 120-cell and the 600-cell are non-facets.*

Thus, with the possible exception of the icosahedron, every regular d-polytope which is not a facet of a *regular* $(d+1)$-polytope is a non-facet. The case of the icosahedron remains open, but note that Theorem 2 implies that there exists no 5-polytope all whose 3-faces are icosahedra. (For a description of the regular polytopes mentioned here the reader may consult [10].)

The following corollary to the proof of Theorem 3 sharpens a result of Schulte [23].

Corollary 4. *There is no face to face tiling of $\mathbf{R}^d$, $d \geq 5$, with crosspolytopes.*

Note that Coxeter [10] constructed a face to face tiling of $\mathbf{R}^4$ with congruent regular crospolytopes.

Motivated by several results asserting the existence of highly symmetrical substructures in "large" structures, we propose

Conjecture 5. *For every two integers ℓ, $k \geq 1$ there is an integer $f(\ell, k)$ such that every d-polytope P, $d \geq f(\ell, k)$, has either an ℓ-dimensional face which is a simplex or a k-dimensional face which is combinatorially isomorphic to a cube.*

We denote by $f(\ell, k)$ the smallest integer satisfying the asserting of the conjecture (if no such integer exists, set $f(\ell, k) = \infty$). Denote by $f_s(\ell, k)$ the corresponding number for *simple* polytopes. Theorem 1 and the 120-cell show that $f(2, 2) = f_s(2, 2) = 5$.

The next result is based on theorems of Nikulin [27,28] and Blind and Blind [7,30]. It asserts that that $f_s(2, k)$ is finite for every k, and moreover, if a simple polytope of sufficiently high dimension does not have a triangular 2-face, then *most* of its k-faces are combinatorially isomorphic to the k-cube.

Theorem 6. *Let $k \geq 1$ be a fixed integer, and let P be a simple d-polytope.*

(i) If $d \geq 2k-1$ then P has a k-face F with fewer than $(k+1)(k+2)$ facets. If $d \geq 2k^2$, then P has a k-face with at most $2k$ facets.

(ii) If P does not have a 2-face which is a triangle and $d \geq 2k^2$ then P has a k-face which is combinatorially isomorphic to the k-cube. Thus $f_s(2, k) \leq 2k^2 < \infty$. If $d > 2k^2 \cdot 1/\varepsilon$ $(0 < \varepsilon < 1)$, then more than $(1-\varepsilon)f_k(P)$ k-faces of P are combinatorially isomorphic to the k-cube. (Here $f_k(P)$ is the total number of k-faces of P.)

2. Face numbers and flag numbers of polytopes

For a d-polytope P and $S \subset \{0, 1, \ldots, d-1\}$, $S = \{i_1, \ldots, i_k\}$ and $i_1 < \ldots < i_k$, the *flag number* $f_S(P)$ is the number of chains $F_1 \subset F_2 \subset \ldots \subset F_k$ of faces of P such that $dimF_\ell = i_\ell$, $k \geq \ell \geq 1$. $f_i(P)$ is the number of i-faces of P. The vector $f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ is called the *f-vector* of P. We will denote by P_i the set of i-faces of P. Put $f_\emptyset = 1$. We will abuse notation and write f_{02} instead of $f_{\{0,2\}}$ etc.

Every d-polytope P satisfies the Euler relation:

$$\sum_{i=0}^{d-1}(-1)^i f_i(P) = 1 - (-1)^d.$$

The set $L(P)$ of all faces of a polytope P, ordered by inclusion, is a lattice called the *face lattice* of P. Every interval $[F, T]$ in $L(P)$ is a face-lattice of a polytope denoted by T/F. (See [15, p.50].) T/F is called the *quotient* of T by F. (It is also sometimes called the link of F in T.) Clearly $dim(T/F) = dimT - dimF - 1$. The lattice obtained by reversing the order relations of $L(P)$ is also a face lattice of a polytope. This polytope is denoted by P^*, and called the *dual* polytope of P. (See [15, p.46].)

Two polytopes are *combinatorially isomorphic* if their face lattices are isomorphic. Since we study in this note only combinatorial properties of polytopes, we will regard, from now on, combinatorially isomorphic polytopes as identical. For example, a d-cube will mean here any polytope combinatorially isomorphic to the d-dimensional cube.

The Euler relations for intervals in the face-lattice of P imply many linear relations for the flag numbers. A complete understanding of the space of linear relations of flag numbers of d-polytopes was achieved by Bayer and Billera [5]. They derived from the Euler relation the following *generalized Dehn-Sommerville* identities:

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup\{j\}} = (1 - (-1)^{k-j-1}) f_S,$$

where S is a subset of $\{0, 1, \ldots, d-1\}$, $\{i, k\} \subset S \cup \{-1, d\}$ and $\{i+1, \ldots, k-1\} \cap S = \emptyset$. Moreover, they showed that these identities span the space of linear relations of flag numbers of d-polytopes.

Define a *special flag number* to be a flag number f_S for a subset S of $\{0, 1, \ldots, d-2\}$, which contains no two consecutive integers. Bayer and Billera proved by successive applications of the generalized Dehn-Sommerville identities, that all flag numbers of d-polytopes can be expressed as linear combinations of special flag numbers. See also [17, 1] and the Appendix for the cases $d = 5$.

For simple polytopes [dually, simplicial polytopes,] the Euler relations for faces [quotients] imply the classic Dehn-Sommerville relations [8, 15, 21] for the face numbers. See Section 5.

Our understanding of linear inequalities which hold for flag numbers of d-polytopes is less complete. For a d-polytope P write $\beta(P) = f_0(P) - d - 1$ and $\gamma(P) = f_1(P) - df_0(P) + \binom{d+1}{2} + f_{02}(P) - 3f_2(P)$.

The fact that every d-polytope has at least $d+1$ vertices is obvious. Thus, $\beta(P) \geq 0$ for every polytope P. A deep inequality is that $\gamma(P) \geq 0$, for every P. For simplicial polytopes this inequality is Barnette's lower bound theorem [3]. The general case is proved by the author in [16] using the rigidity theory of frameworks.

Define a *d-form* to be a linear combination of special flag numbers of d-polytopes. A d-form m is nonnegative if $m(P) \geq 0$ for every d-polytope P. Given a d-form m_1 and an e-form m_2, define their *convolution* $m_1 * m_2$, as the unique $(d+e+1)$-form which satisfies $m_1 * m_2(P) = \Sigma\{m_1(F)m_2(P/F) : F \in P_d\}$, for every $(d+e+1)$-polytope P. See [17, Sec.2.]. Clearly, if m_1 and m_2 are nonnegative forms then so is $m_1 * m_2$.

Put $\beta^*(P) = \beta(P^*)$ and $\gamma^*(P) = \gamma(P^*)$. The nonnegative forms $1, \beta, \beta^*, \gamma$ and γ^* generate by convolutions many further nonnegative forms. The 21 resulting inequalities for flag numbers of 5-polytopes are given in the Appendix. See also [1] (for the 4-dimensional case) and [17, Sec. 6].

Remark. β and γ are the first two in a sequence of $[d/2]$ linear combinations of flag numbers of d-polytopes which are conjectured to be non-negative for all d-polytopes, and believed to be crucial invariants in the combinatorial theory of polytopes. See [17, 26].

3. Faces of 5-polytopes

Proof of Theorem 1. Let P be a 5-polytope all whose 2-faces have at least five vertices. Recall that P_k denotes the set of k-faces of P. Consider the following weights on P_2. For $F \in P_2$, define $w(F) = f_0(P/F) - 2$ and $W = \Sigma\{w(F) : F \in P_2\}$. Note that $w(F) \geq 1$ for every F.

Theorem 1 follows from the following stronger assertion.

Theorem 7. *For every 5-polytope P,*

$$(*) \qquad \Sigma\{f_0(F) \cdot w(F) : F \in P_2\} < 5W.$$

Proof. We will consider the following inequalities for P. Each inequality is first given in terms of β and γ, then in terms of flag numbers and finally in terms of special flag numbers.

1) $\beta^* = f_4 - 6 = f_3 - f_2 + f_1 - f_0 - 4 \geq 0$.

2) $1/2\Sigma\{\beta^*(F/G) : F \in P_4, G \in P_0\} = 1/2(f_{014} - 4f_{04}) = f_{13} - 2f_{03} + f_{02} - 2f_1 \geq 0$.

3) $\Sigma\{\gamma(P/F) : F \in P_0\} = f_{02} + f_{013} - 3f_{03} - 4f_{01} + 10f_0 = f_{02} + 2f_{13} - 3f_{03} - 8f_1 + 10f_0 \geq 0$

If $(*)$ is violated we must also have

4) $\Sigma\{(f_0(F) - 5) \cdot (f_0(P/F) - 2) : F \in P_2\} = f_{024} - 2f_{02} - 5f_{24} + 10f_2 = 5f_{03} - 3f_{13} - 2f_{02} - 10f_3 + 10f_2 \geq 0$.

Adding 2, 3 and 4 one gets $10f_0 - 10f_1 + 10f_2 - 10f_3 = 20 - 10f_4 \geq 0$ which contradicts 1. ∎

The proof of Theorem 1 for the special case of simple polytopes is much simpler. The Dehn-Sommerville relations (see [15, p. 425],) give for a simple 5-polytope P,

$$f_{12} - 5f_2 = 4f_1 - 5f_2 = 20 - 10f_4 < 0.$$

Thus, the average number of vertices of 2-faces of a simple 5-polytope is strictly less than five. In a dual form, we obtain that if C is a simplicial 4-dimensional sphere, than the average number of vertices in the links of 2-faces of C is strictly less than five. This implies that every locally-finite simplicial or cubical d-manifold (with no boundary) of dimension five or more has a $(d-2)$-face L whose link is $\triangle$ or $\square$.

Gromov ([14], Sec. 4) described a natural metric on cubed manifolds, and introduced a combinatorial condition for the curvature to be negative. This "no $\triangle$ no $\square$" condition implies that the link of every $(d-2)$-face is neither a $\triangle$ nor a $\square$. If follows that Gromov's condition can never be satisfied for d-manifolds, $d \geq 5$.

The proof of Theorem 1 was obtained as follows: We wrote all 21 linear inequalities for the flag numbers of 5-polytopes which can be derived from β, β^*, γ and γ^*. We used a computer to solve the linear program obtained by adding inequality 4) to these inequalities, and got the answer that there is no solution. I am thankful to Edna Wigdorson for her valuable help in this part of the work.

Let us remark that the inequalities for flag numbers of 5-polytopes do *not* imply any absolute bound on the average number of vertices of 2-faces in a 5-polytope. It is an interesting open problem to find such a bound, or better (in view of Conjecture A in [17, Sec. 6]), to construct 5-polytopes with arbitrary large average size of 2-faces.

Theorem 2 follows from the following stronger assertion.

Theorem 8. *The average number of vertices in $F/\{v\}$ over all 3-faces F and vertices $v \in F$ is strictly less than 4.5.*

Proof. If the assertion of Theorem 8 is false then (**) $f_{013}-4.5f_{03} = 2f_{13}-4.5f_{03} \geq 0$. The linear program obtained by adding (**) to our 21 inequalities didn't have any feasible solution. ∎

For the contradiction in the proof of Theorem 1, the inequality $\gamma \geq 0$ is needed. It is not known whether this inequality holds for non-polytopal spheres. (I conjecture that γ is non-negative for every polyhedral pseudomanifold [17, Sec. 9].) The proof of Theorem 8 relies *only* on those inequalities derived from $\beta \geq 0$. Theorem 8 thus applies to arbitrary Eulerian lattices.

4. Facets and non-facets

Proof of Theorem 3. In the case where all facets of a 5-polytope P are (combinatorially) regular polytopes, it is easy to express all flag numbers of P as linear combinations of two face numbers. Let P be a 5-polytope all whose 4-faces are 4-crosspolytopes. Then, $2f_3 = 16f_4$ and $2f_1 - 3f_2 + 4f_3 - 8f_4 = 0$. Setting $a = f_4$ and $b = f_3$, the f-vector of P is $(\frac{1}{2}b - 5a + 2, \frac{3}{2}b - 12a, b, 8a, a)$. Also we have $f_{02} = 3b, f_{03} = 32a$, and $f_{13} = 48a$. It turns out that these relations are not compatible with the inequalities for flag-numbers of 5-polytopes.

Thus, the 4-crosspolytope is a non-facet, and this implies that the d-crosspolytope is a non-facet for every $d \geq 4$. (For $d \geq 6$ this was proved already in [22].) The situation for the other cases of Theorem 3 is completely the same. ∎

The proof of Theorem 3 relies only on those inequalities derived from $\beta \geq 0$. Theorem 3 thus applies to arbitrary Eulerian lattices. In particular, Theorem 3 applies to spherical polytopes and can be used to identify non-tiles as done by Schulte [23].

Proof of Corollary 4. Let U be a face to face tiling of $\mathbb{R}^d$ by d-crosspolytopes. The vertex figure of a vertex of U is a spherical d-polytope all whose facets are combinatorially isomorphic to the $(d-1)$-crosspolytope. This is impossible for $d \geq 5$. ∎

In a similar way to the above proofs one proves that in every quasi-simplicial 5-polytope, the average value of $\gamma(F)$ on F is strictly less than 5. This gives a new proof to the fact that every simplical 4-polytope P satisfying $\gamma(P) \geq 5$ is a non-facet ([22,(50)]). It follows that the class of facet-forming simplical 4-polytopes is nowhere dense in the class of 4-dimensional convex bodies. (It is known that γ tends to infinity for every sequence of simplical polytopes which converges to a smooth convex body. [16].)

5. Low-dimensional faces of simple polytopes

Let P be a simple d-polytope. Consider the vector $h = (h_0, h_1, \ldots, h_d)$, defined by the relations

$$f_k = h_k + \binom{k+1}{k} h_{k+1} + \ldots + \binom{k+j}{k} h_{k+j} + \ldots + \binom{d}{k} h_d, \; k = 0, 1, \ldots, d.$$

This vector is the *h-vector* of the dual polytope P^*.

The Dehn-Sommerville relations assert that $h_i = h_{d-i}$, for every i. ($h_0 = h_d$ is the Euler relation). Another fundamental property of the h-numbers of simple polytopes is that $h_i \geq 0$ for every i. See [21,8].

We also need the following result of Blind and Blind [7].

Theorem BB. *Let P be a simple d-polytope with no triangular 2-faces. Then (i) P has at least $2d$ facets, (ii) if P has exactly $2d$ facets then P is combinatorically isomorphic to the d-cube.*

Remark. Theorem BB was proved in respond to a conjecture of Kupitz [18], asserting that if P is a d-polytope and P has no triangular 2-faces, then $f_i(P) \geq f_j(C_d)$, for every j. (Here, C_d is the d-cube.) Kupitz also conjectured that equality holds only for the d-cube. Blind and Blind settled the conjecture for simple polytopes and in several other cases. (Added in proof: Kupitz's conjecture was recently proved by Blind and Blind [30].)

Proof of Theorem 6. We first consider the case where d is odd. Put $d = 2r+1$. Let $k \leq r$.

$$f_k = \left[\binom{r}{k} + \binom{r+1}{k}\right] h_r + \left[\binom{r-1}{k} + \binom{r+2}{k}\right] h_{r-1} + \ldots + \left[\binom{r-j}{k} + \binom{r+j+1}{k}\right] h_{r-j} + \ldots.$$

Since P is simple, $f_{k-1,k} = f_{k-1}(d-k+1)$ for every k. By a comparison of the representations of f_r and f_{r+1} as linear combinations of h-numbers, it is easy to see that $f_r - (r+2)f_{r+1}$ is a positive combination of $h_{r+2}, \ldots, h_d$. Therefore $(r+1)(r+2)f_{r+1} > (r+1)f_r = f_{r,r+1}$. Thus, the average number of facets in an $r+1$-face of P is less than $(r+1)(r+2)$. For d even the proof is similar. Consider the expression $u_k = (2k+\varepsilon)f_k - (d-k+1)f_{k-1}$. A simple calculation shows that u_k is a positive combination of h-numbers, for $d > 2 \cdot k^2 \cdot 1/\varepsilon$. This proves (i). (See note added in proof.)

Let P be a d-polytope with no triangular 2-face. If $d \geq 2k^2$, then P must have a k-face F with at most $2k$ facets. Since F itself has no triangular 2-faces, it follows from Theorem BB(i) that F has at least $2k$ facets. Therefore F has exactly $2k$ facets, and by Theorem BB(ii), F is a cube. If $d \geq 2k^2 1/\varepsilon$, then at least $(1-\varepsilon)f_k$ k-faces of P has $2k$ facets each, so at least $(1-\varepsilon)f_k$ k-faces of P are cubes. This completes the proof of Theorem 6. ∎

We cannot extend Theorem 6(iii) to simple polytopes with no simplex as an ℓ-face, for $\ell \geq 3$. There is, however, a natural class of polytopes for which this can be done. Let $r = (r_1, r_2, \ldots, r_t)$ be a partition of d. A simplical d-polytope P is *r-balanced* if its vertices can be colored with t colors so that every facet has r_i vertices of color i. See [25,6]. If $r_i \leq \ell$ for every i then P^* does not have an ℓ-face which is a simplex. In this case the existence of triangle free faces of dimension $> d/(\ell-1)$ is evident, and it follows from Theorem 6 that if $d \gg k, \ell$ then P^* has a k-face which is a cube. In fact, using the extension of the Dehn-Sommerville relations for such polytopes [6], it can be proved that if d is sufficiently large, most k-dimensional faces of P^* are cubes.

The proof of Theorem BB(i) extends directly to show that a simple d-polytope with no simplex as an ℓ-face, has at least $\ell d/(\ell-1)$ facets. Using this, it can be proved that for such polytopes, the average number of facets in a k-face is close to $2k$ when $d \gg k$.

6. Final remarks

1. The fact that the sum of the vertex degrees of the graph of a 3-polytope with n vertices is at most $6n-12$ is stated already in René Descartes' work on polyhedra. In Descartes' words (translated to English by Federico, [12, p.57, Prop. 6]): "I always take α for the number of solid angels... The actual number of plane angels... cannot exceed $6\alpha - 12$."

2. Is it true, in some sense that a typical k-face of a typical simple d-polytope, $d \gg k$, is combinatorially isomorphic to the k-cube?

3. It was asked in [22]: "If P is a simple d-polytope which is facet-forming, is there a simple $(d+1)$-polytope Q all whose facets are isomorphic to P?" The following example shows that the answer is negative for $d \geq 4$. Let P be a prism over a $d-1$-simplex, $d \geq 4$. P is facet-forming ([22]). Note that P is dual to a stacked polytope. If Q is a simple $(d+1)$-polytope all whose d-facets are isomorphic to P, then by [16], Q is dual to a stacked polytope, but then Q has a facet which is a simplex.

4. Conjecture 5 is known for zonotopes. McMullen [20] proved that every d-zonotope has a $[(d+1)/2]$-face which is a parallelotope. There is a simple correspondence between d-zonotopes and arrangements of points in the projective $(d-1)$-space (see, e.g., [20, Sec. 7]). Under this correspondence, the fact that every zonotope has a 2-face which is a quadrilateral, is just the Sylvester-Gallai theorem (which thus immediately follows from Theorem 0, as is well known). McMullen's result for $d \geq 4$

follows from Hansen's high dimensional extension of the Sylvester-Gallai theorem [13].

It is amusing to check Conjecture 5 for various polytopes based on combinatorial structures. For the vertex-packing polytope (see e.g., [9]), Ramsey's theorem implies the existence of k-faces (of a very special nature) which are simplices or cubes.

Problems analogous to Conjecture 5 may be studied for other classes of graded posets, such as incidence polytopes and geometric lattices.

5. Define a *regular sequence* of polytopes $\{P_n : n \geq 0\}$ by the property that each P_n is an n-polytope and each k-face of P is combinatorically isomorphic to P_k. The only regular sequences of simple polytopes are the sequences of simplices and of cubes. (If conjecture 5 is true, there are no other regular sequences of (arbitrary) polytopes.) Define a *semi-regular* sequence of polytopes $\{P_n : n \geq 0\}$, $\dim P_n = n$, by the property that each face of each P_n is combinatorically isomorphic to a Cartesian product of P_i's. Two existing semi-regular sequences of simple polytopes are the permutahedra (duals to the first barycentric subdivision of simplices,) and associahedra [19]. A general study of such sequences would be of interest.

Appendix: linear relations for flag-numbers of 5-polytopes.

1.Linear equalities. There are 32 flag-numbers and they can be expressed as linear combinations of the special flag-numbers $1(= f_\emptyset)$, f_0, f_1, f_2, f_3, f_{02}, f_{03} and f_{13}. We express those flag-numbers which are needed for the linear inequalities below. $f_4 = f_3 - f_2 + f_1 - f_0 + 2$, $f_{01} = 2f_1$, $f_{34} = 2f_3$, $f_{04} = f_{03} - f_{02} + 2f_1$, $f_{12} = f_{02}$, $f_{14} = f_{13} - f_{12} + 2f_1$, $f_{23} = f_{24} = f_{13} + f_{03} - 2f_3$, $f_{123} = f_{124} = f_{013} = f_{024} = 2f_{13}$, $f_{034} = 2f_{03}$, etc.

2. Linear inequalities. We make the following notation. $\beta_{i,j} = \Sigma\{T/F) : T \in P_j,\ F \in P_i\}$, and $\beta_j = \beta_{-1,j} = \Sigma\{\beta(T) : T \in P_j\}$. The same notation applies to β^* and γ.

Let P be a d-polytope. Note that for $d < 2, \beta(P) = 0$; for $d = 2, \beta(P) = \beta^*(P)$; for $d < 4, \gamma(P) = 0$ and for $d = 4, \gamma(P) = \gamma^*(P)$.

1. $\beta = f_0 - 6 \geq 0$
2. $\beta^* = f_4 - 6 = f_3 - f_2 + f_1 - f_0 - 4 \geq 0$
3. $\gamma = f_{02} - 3f_2 + f_1 - 5f_0 + 15 \geq 0$
4. $\gamma^* = f_{24} - 3f_2 + f_3 - 5f_4 + 15 = f_{13} - f_{03} - 2f_3 + 2f_2 - 5f_1 + 5f_0 + 5 \geq 0$
5. $\beta_4 = f_{04} - 5f_4 = f_{03} - f_{02} - 5f_3 + 5f_2 - 3f_1 + 5f_0 - 10 \geq 0$
6. $\beta_4^* = f_{34} - 5f_4 = -3f_3 + 5f_2 - 5f_1 + 5f_0 - 10 \geq 0$
7. $\gamma_4 = f_{024} - 3f_{24} + f_{14} - 4f_{04} + 10f_4 = -f_{03} + 3f_{02} + 4f_3 - 10f_2 + 4f_1 - 10f_0 + 20 \geq 0$
8. $\beta_{0,5} = f_{01} - 5f_0 = 2f_1 - 5f_0 \geq 0$
9. $\beta_{0,5}^* = f_{04} - 5f_0 = f_{03} - f_{02} + 2f_1 - 5f_0 \geq 0$
10. $\gamma_{0,5} = f_{013} - 3f_{03} + f_{02} - 4f_{01} + 10f_0 = 2f_{13} - 3f_{03} + f_{02} - 8f_1 + 10f_0 \geq 0$
11. $\beta_3 = f_{03} - 4f_3 \geq 0$
12. $\beta_3^* = f_{23} - 4f_3 = f_{13} - f_{03} - 2f_3 \geq 0$
13. $\beta_{0,4} = f_{014} - 4f_{04} = 2f_{13} - 4f_{03} + 2f_{02} - 4f_1 \geq 0$
14. $\beta_{0,4}^* = f_{034} - 4f_{04} = -2f_{03} + 4f_{02} - 8f_1 \geq 0$
15. $\beta_{1,5} = f_{12} - 4f_1 = f_{02} - 4f_1 \geq 0$
16. $\beta_{1,5}^* = f_{14} - 4f_1 = f_{13} - f_{02} - 2f_1 \geq 0$

17. $\beta_2 = f_{02} - 3f_2 \geq 0$
18. $\beta_{0,3} = f_{013} - 3f_{03} = 2f_{13} - 3f_{03} \geq 0$
19. $\beta_{1,4} = f_{124} - 3f_{14} = -f_{13} + 3f_{02} - 6f_1 \geq 0$
20. $\beta_{2,5} = f_{23} - 3f_2 = f_{13} - f_{03} + 2f_3 - 3f_2 \geq 0$
21. $\beta_2 * \beta_{2,5} = \Sigma\{\beta(F)\beta(P/F) : F \in P_2\} = f_{024} - 3f_{02} - 3f_{24} + 9f_2 = -f_{13} + 3f_{03} - 3f_{02} - 6f_3 + 9f_2 \geq 0$

Note added in proof:

Part (i) in Theorem 6 follows from the following theorem of V. V. Nikulin [27,28]. (See also Khovanskii [29].)

Theorem N. *The average number of l-dimensional faces of a k-dimensional face of a simple n-dimensional polytope for $0 \leq \ell < k < (d+1)/2$ is at most*

$$\binom{n-\ell}{n-k}\left(\binom{[n/2]}{\ell}+\binom{[(n+1)/2]}{\ell}\right)\Big/\left(\binom{[n/2]}{k}+\binom{[(n+1)/2]}{k}\right).$$

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